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2002 J. Phys. A: Math. Gen. 35 4363

(http://iopscience.iop.org/0305-4470/35/19/316)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **35** (2002) 4363–4373 PII: S0305-4470(02)33351-1

Difference L operators and a Casorati determinant solution to the T-system for twisted quantum affine algebras

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Received 30 January 2001
Published 3 May 2002
Online at stacks.iop.org/JPhysA/35/4363

Abstract
We propose factorized difference operators $L(u)$ associated with the twisted quantum affine algebras $U_q(A_{2n}^{(2)}), U_q(A_{2n-1}^{(2)}), U_q(D_{n+1}^{(2)}), U_q(D_4^{(3)})$. These operators are shown to be annihilated by a screening operator. Based on the solutions of the difference equation $L(u)w(u) = 0$, we also construct a Casorati determinant solution to the T-system for $U_q(A_{2n}^{(2)}), U_q(A_{2n-1}^{(2)})$.

PACS numbers: 05.50.+q, 02.10.−v

1. Introduction

In [1], a class of functional relations, a T-system, was proposed for commuting transfer matrices of solvable lattice models associated with twisted quantum affine algebras $U_q(X_N^{(r)})$ $(r > 1)$. For $X_N^{(r)} = A_N^{(2)}$, it has the following form.

For the $U_q(A_{2n}^{(2)})$ case:

$$T_m^{(a)}(u-1)T_m^{(a)}(u+1) = T_{m-1}^{(a)}(u)T_{m+1}^{(a)}(u) + T_m^{(a-1)}(u)T_m^{(a+1)}(u) \qquad \text{for} \quad 1 \leqslant a \leqslant n-1$$
$$T_m^{(n)}(u-1)T_m^{(n)}(u+1) = T_{m-1}^{(n)}(u)T_{m+1}^{(n)}(u) + T_m^{(n-1)}(u)T_m^{(n)}\left(u+\frac{\pi \mathrm{i}}{2\hbar}\right). \tag{1.1}$$

For the $U_q(A_{2n-1}^{(2)})$ case:

$$T_m^{(a)}(u-1)T_m^{(a)}(u+1) = T_{m-1}^{(a)}(u)T_{m+1}^{(a)}(u) + T_m^{(a-1)}(u)T_m^{(a+1)}(u) \qquad \text{for} \quad 1 \leqslant a \leqslant n-1$$
$$T_m^{(n)}(u-1)T_m^{(n)}(u+1) = T_{m-1}^{(n)}(u)T_{m+1}^{(n)}(u) + T_m^{(n-1)}(u)T_m^{(n-1)}\left(u+\frac{\pi \mathrm{i}}{2\hbar}\right). \tag{1.2}$$

Here $\{T_m^{(a)}(u)\}_{a\in I_\sigma; m\in\mathbb{Z}_{\geqslant 1}; u\in\mathbb{C}}$ $(I_\sigma = \{1, 2, \ldots, n\})$ are the transfer matrices with the auxiliary space labelled by a and m. We shall adopt the boundary condition $T_{-1}^{(a)}(u) = 0$, $T_0^{(a)}(u) = 1$, which is natural for the transfer matrices. This T-system (1.1), (1.2) is a kind of discrete Toda equation, which follows from a reduction of the Hirota–Miwa equation [2, 3]. The original T-system [1] contains a scalar function $g_m^{(a)}(u)$ in the second term of the rhs of (1.1), (1.2). Throughout this paper, we set $g_m^{(a)}(u) = 1$. This corresponds to the case where the vacuum part is formally trivial. However, the structure of the solution of (1.1), (1.2) is essentially independent of the function $g_m^{(a)}(u)$. In this paper, we briefly report on a new expression for the solution of (1.1), (1.2) motivated by the recently found interplay [4] between factorized difference L operators and the q-characters for non-twisted quantum affine algebras [5, 6].

In section 2, we propose factorized difference operators $L(u)$ for $U_q\big(A_{2n}^{(2)}\big)$, $U_q\big(A_{2n-1}^{(2)}\big)$, $U_q\big(D_{n+1}^{(2)}\big)$, $U_q\big(D_4^{(3)}\big)$. $L(u)$ generates functions $\{T^a(u)\}_{a\in\mathbb{Z};u\in\mathbb{C}}$, which are Laurent polynomials in variables $\{Y_a(u)\}_{a\in I_\sigma;u\in\mathbb{C}}$. Moreover $Y_a(u)$ is expressed by a function $Q_a(u)$ which corresponds to the Baxter Q-function. When $Q_a(u)$ is suitably chosen in the context of the analytic Bethe ansatz [1, 7–9], $T^a(u)$ corresponds to an eigenvalue formula of the transfer matrix in the dressed vacuum form (DVF). In particular for $1 \leqslant a \leqslant b\big(U_q\big(A_{2n}^{(2)}\big), U_q\big(A_{2n-1}^{(2)}\big)\colon\ b = n;\ U_q\big(D_{n+1}^{(2)}\big)\colon b = n-1;\ U_q\big(D_4^{(3)}\big)\colon\ b = 2\big)$, the auxiliary space for this transfer matrix is expected [1] to be a finite-dimensional irreducible module of the quantum affine algebra [10, 11], which is called the Kirillov–Reshetikhin module $W_1^{(a)}(u)$ (see also section 5 in [12]). One of the intriguing properties of $L(u)$ is that $L(u)$ is annihilated by a screening operator $\{\mathcal{S}_a\}_{a\in I_\sigma}$, from which $\big(\mathcal{S}_a \cdot T^a\big)(u) = 0$ results. In the context of the analytic Bethe ansatz, this corresponds to the pole-freeness of $T^a(u)$ under the Bethe ansatz equation. For the non-twisted case $U_q\big(X_N^{(1)}\big)$, one may identify $\mathcal{S}_a$ with the Frenkel–Reshetikhin screening operator [5] if $Q_a(u)$ is suitably chosen.

For the $U_q\big(A_N^{(2)}\big)$ case, $L(u)$ becomes of the order of $N+1$. By using a basis of the solutions of the difference equation $L(u)w(u) = 0$, in section 3, we give a solution (theorem 3.6) of the T-system for $U_q\big(A_N^{(2)}\big)$ (1.1), (1.2) as a ratio of two Casorati determinants whose matrix size is constantly $(N+1)\times(N+1)$. On solving this T-system, a duality relation (proposition 2.7) plays an important role. There is another expression of the solution to the $U_q\big(A_N^{(2)}\big)$ T-system (1.1), (1.2) which is described by semi-standard tableaux with rectangular shape [1]. This solution follows from a reduction of the Bazhanov and Reshetikhin's Jacobi–Trudi type formula [13] (see (3.4)). In contrast to the Casorati determinants case, the size of the matrix for this determinant is $m \times m$ and thus increases as m increases. Lemma 3.3 connects these two types of solutions.

In contrast to the $U_q\big(A_N^{(2)}\big)$ case, $L(u)$ for $U_q\big(D_{n+1}^{(2)}\big)$, $U_q\big(D_4^{(3)}\big)$ contain factors which have a negative exponent -1, thus their order become infinite. Therefore we cannot straightforwardly extend the analysis to get the Casorati determinant type solution for $U_q\big(A_N^{(2)}\big)$ in this case. However Jacobi–Trudi type formulae are still available in this case as reductions of the solutions in [14, 15]. This situation is parallel to the non-twisted $U_q\big(D_n^{(1)}\big)$ case [4].

The deformation parameter q is expressed by a parameter $\hbar$ as $q = \mathrm{e}^{\hbar}$. The parameter $\hbar$ often appears as a multiple of $\frac{\pi \mathrm{i}}{r\hbar}$. However, we note that our argument in this paper is also valid even if one formally sets $\frac{\pi \mathrm{i}}{r\hbar} = 0$. In this case, the T-system (1.1) is equivalent to the one for the superalgebra $B^{(1)}(0|n)$ [16].

In this paper, we omit most of the calculations and proofs, which are parallel with those in the non-twisted case [4].

2. Difference L operators

Let X_N be a complex simple Lie algebra of rank N, σ a Dynkin diagram automorphism of X_N of order $r = 1, 2, 3$. The affine Lie algebras of type $X_N^{(r)} = A_n^{(1)}$ $(n \geqslant 1)$, $B_n^{(1)}$ $(n \geqslant 2)$,

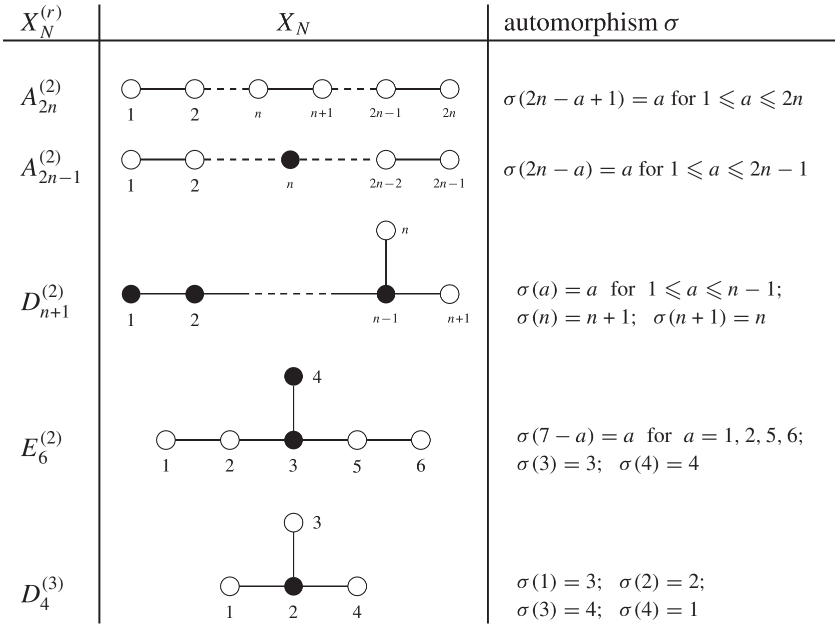

Figure 1. The Dynkin diagrams of X_N for $r>1$: The enumeration of the nodes with I specified under or on the right side of the nodes. The filled circles denote the fixed points of the Dynkin diagram automorphism σ of order r.

$C_n^{(1)}$ $(n\geqslant 2)$, $D_n^{(1)}$ $(n\geqslant 4)$, $E_n^{(1)}$ $(n=6,7,8)$, $F_4^{(1)}$, $G_2^{(1)}$, $A_{2n}^{(2)}$ $(n\geqslant 1)$, $A_{2n-1}^{(2)}$ $(n\geqslant 2)$, $D_{n+1}^{(2)}$ $(n\geqslant 2)$, $E_6^{(2)}$ and $D_4^{(3)}$ are realized as the canonical central extension of the loop algebras based on the pair (X_N,σ). We write the set of the nodes of the Dynkin diagram of X_N as $I=\{1,2,\ldots,N\}$, and let $I_\sigma=\{1,2,\ldots,n\}$ be the set of σ-orbits of I. In particular, $N=n$ and $I=I_\sigma$ for the non-twisted case $r=1$. We define numbers $\{r_a\}_{a\in I}$ such that $r_a=r$ if $\sigma(a)=a$, otherwise $r_a=1$. In our enumeration of the notes of the Dynkin diagram (see figure 1), r_a is 1 except for the case: $r_n=2$ for $A_{2n-1}^{(2)}$, $r_a=2$ $(1\leqslant a\leqslant n-1)$ for $D_{n+1}^{(2)}$, $r_3=r_4=2$ for $E_6^{(2)}$, $r_2=3$ for $D_4^{(3)}$. Let $\{\alpha_a\}_{a\in I}$ be the simple roots of X_N with a bilinear form $(\cdot|\cdot)$ normalized as $(\alpha|\alpha)=2$ for a long root α. Let I_{ab} be an element of the incidence matrix of X_N: $I_{ab}=2\delta_{ab}-2(\alpha_a|\alpha_b)/(\alpha_a|\alpha_a)$.

Let $U_q\big(X_N^{(r)}\big)$ be the quantum affine algebra. We introduce functions $\{Q_a(u)\}_{a\in I_\sigma;u\in\mathbb{C}}$ which correspond to the Baxter Q functions for $U_q\big(X_N^{(r)}\big)$, and define functions $\{Y_a(u)\}_{a\in I_\sigma;u\in\mathbb{C}}$ as

$$Y_a(u)=\frac{Q_a\left(u-\frac{1}{2}(\alpha_a|\alpha_a)\right)}{Q_a\left(u+\frac{1}{2}(\alpha_a|\alpha_a)\right)}. \tag{2.1}$$

We formally set $Y_0(u)=1$; $Q_{n+1}(u)=Q_n\big(u+\frac{\pi\mathrm{i}}{2\hbar}\big)$ and $Y_{n+1}(u)=Y_n\big(u+\frac{\pi\mathrm{i}}{2\hbar}\big)$ for $X_N^{(r)}=A_{2n}^{(2)}$; $Q_{n+1}(u)=1$ and $Y_{n+1}(u)=1$ for $X_N^{(r)}\neq A_{2n}^{(2)}$. For the twisted case $r>1$, we assume quasi-periodicity $Q_a\big(u+\frac{\pi\mathrm{i}}{\hbar}\big)=h_aQ_a(u)$ $(h_a\in\mathbb{C})$, which induces periodicity $Y_a\big(u+\frac{\pi\mathrm{i}}{\hbar}\big)=Y_a(u)$. For the non-twisted case $r=1$, one can identify $Y_a(u)$ with the Frenkel–Reshetikhin variable Y_{a,q^u} [5] denoted as $Y_a(u)$ in [4] if $Q_a(u)$ is suitably chosen. We shall also use notations $Q_a^k(u)=\prod_{j=0}^{k-1}Q_a\big(u+\frac{\pi\mathrm{i}j}{r\hbar}\big)$ and $Y_a^k(u)=\prod_{j=0}^{k-1}Y_a\big(u+\frac{\pi\mathrm{i}j}{r\hbar}\big)$.

Next we introduce screening operators $\{\mathcal{S}_a\}_{a\in I_\sigma}$ on $\mathbb{Z}\big[Y_a(u)^{\pm 1}\big]_{a\in I_\sigma; u\in\mathbb{C}}$, whose action is given by

$$(\mathcal{S}_a \cdot Y_b)(u) = \delta_{ab} Y_a(u) S_a(u). \tag{2.2}$$

Here we assume $S_a(u)$ satisfies the following relation:

$$S_a(u + (\alpha_a|\alpha_a)) = A_a\left(u + \tfrac{1}{2}(\alpha_a|\alpha_a)\right) S_a(u) \tag{2.3}$$

$$A_a(u) = \prod_{b=1}^{n'} \frac{Q_b^{r_{ab}}(u - (\alpha_a|\alpha_b))}{Q_b^{r_{ab}}(u + (\alpha_a|\alpha_b))} \tag{2.4}$$

where $r_{ab} = \max(r_a, r_b)$; $n' = n+1$ for $X_N^{(r)} = A_{2n}^{(2)}$ and $n' = n$ for $X_N^{(r)} \neq A_{2n}^{(2)}$. We assume $\mathcal{S}_a$ obeys the Leibniz rule. The origin of (2.4) goes back to the Reshetikhin and Wiegmann's Bethe ansatz equation [17] (cf (4.1)). For the non-twisted $r = 1$ case, (2.4) reduces to the corresponding equation in [4]. We have a formal solution of (2.3) (see also section 5 in [5]):

$$S_a(u) = \frac{\prod_{b=1}^{n'} K_{ab}(u)}{Q_a^{r_a}\left(u - \frac{1}{2}(\alpha_a|\alpha_a)\right) Q_a^{r_a}\left(u + \frac{1}{2}(\alpha_a|\alpha_a)\right)} \tag{2.5}$$

where

$$K_{ab}(u) = \begin{cases} 1 & \text{if} \quad I_{ab} = 0 \\ Q_b^{r_{ab}}(u) & \text{if} \quad I_{ab} = 1 \\ Q_b\left(u - \frac{1}{2}\right) Q_b\left(u + \frac{1}{2}\right) & \text{if} \quad I_{ab} = 2 \\ Q_b\left(u - \frac{2}{3}\right) Q_b(u) Q_b\left(u + \frac{2}{3}\right) & \text{if} \quad I_{ab} = 3. \end{cases} \tag{2.6}$$

Owing to the Leibniz rule, we have

$$\left(\mathcal{S}_a \cdot Y_b^k\right)(u) = \delta_{ab} Y_a^k(u) \sum_{j=0}^{k-1} S_a\left(u + \frac{\pi \mathrm{i} j}{r\hbar}\right). \tag{2.7}$$

We shall use the following variables for each algebra; the origin of these variables goes back to the analytic Bethe ansatz calculation of DVF [1, 8, 9].

For the $U_q\left(A_{2n}^{(2)}\right)$ case:

$$\begin{aligned} z_a(u) &= \frac{Y_a(u+a)}{Y_{a-1}(u+a+1)} \qquad \text{for} \quad 1 \leqslant a \leqslant n \\ z_0(u) &= \frac{Y_n\left(u+n+1+\frac{\pi\mathrm{i}}{2\hbar}\right)}{Y_n(u+n+2)} \\ z_{\bar{a}}(u) &= \frac{Y_{a-1}\left(u+2n-a+2+\frac{\pi\mathrm{i}}{2\hbar}\right)}{Y_a\left(u+2n-a+3+\frac{\pi\mathrm{i}}{2\hbar}\right)} \qquad \text{for} \quad 1 \leqslant a \leqslant n. \end{aligned} \tag{2.8}$$

We also use the variables: $x_a(u) = z_a(u)$ and $x_{2n-a+2}(u) = z_{\bar{a}}(u)$ for $1 \leqslant a \leqslant n$; $x_{n+1}(u) = z_0(u)$.

For the $U_q\left(A_{2n-1}^{(2)}\right)$ case:

$$\begin{aligned} z_a(u) &= \frac{Y_a(u+a)}{Y_{a-1}(u+a+1)} \qquad \text{for} \quad 1 \leqslant a \leqslant n-1 \\ z_n(u) &= \frac{Y_n^2(u+n)}{Y_{n-1}(u+n+1)} \\ z_{\bar{n}}(u) &= \frac{Y_{n-1}\left(u+n+1+\frac{\pi\mathrm{i}}{2\hbar}\right)}{Y_n^2(u+n+2)} \\ z_{\bar{a}}(u) &= \frac{Y_{a-1}\left(u+2n-a+1+\frac{\pi\mathrm{i}}{2\hbar}\right)}{Y_a\left(u+2n-a+2+\frac{\pi\mathrm{i}}{2\hbar}\right)} \qquad \text{for} \quad 1 \leqslant a \leqslant n-1. \end{aligned} \tag{2.9}$$

We also use the variables: $x_a(u) = z_a(u)$ and $x_{2n-a+1}(u) = z_{\bar{a}}(u)$ for $1 \leqslant a \leqslant n$.

For the $U_q\left(D_{n+1}^{(2)}\right)$ case:

$$\begin{aligned}
z_a(u) &= \frac{Y_a^2(u+a)}{Y_{a-1}^2(u+a+1)} \qquad \text{for} \quad 1 \leqslant a \leqslant n \\
z_{n+1}(u) &= \frac{Y_n\left(u+n+\frac{\pi \mathrm{i}}{2\hbar}\right)}{Y_n(u+n+2)} \\
z_{\overline{n+1}}(u) &= \frac{Y_n(u+n)}{Y_n\left(u+n+2+\frac{\pi \mathrm{i}}{2\hbar}\right)} \\
z_{\bar{a}}(u) &= \frac{Y_{a-1}^2(u+2n-a+1)}{Y_a^2(u+2n-a+2)} \qquad \text{for} \quad 1 \leqslant a \leqslant n.
\end{aligned} \tag{2.10}$$

For the $U_q\left(D_4^{(3)}\right)$ case:

$$\begin{aligned}
z_1(u) &= Y_1(u+1) \\
z_2(u) &= \frac{Y_2^3(u+2)}{Y_1(u+3)} \\
z_3(u) &= \frac{Y_1^3(u+3)}{Y_1(u+3)Y_2^3(u+4)} \\
z_4(u) &= \frac{Y_1\left(u+3-\frac{\pi \mathrm{i}}{3\hbar}\right)}{Y_1\left(u+5+\frac{\pi \mathrm{i}}{3\hbar}\right)} \\
z_{\bar{4}}(u) &= \frac{Y_1\left(u+3+\frac{\pi \mathrm{i}}{3\hbar}\right)}{Y_1\left(u+5-\frac{\pi \mathrm{i}}{3\hbar}\right)} \\
z_{\bar{3}}(u) &= \frac{Y_1(u+5)Y_2^3(u+4)}{Y_1^3(u+5)} \\
z_{\bar{2}}(u) &= \frac{Y_1(u+5)}{Y_2^3(u+6)} \\
z_{\bar{1}}(u) &= \frac{1}{Y_1(u+7)}.
\end{aligned} \tag{2.11}$$

Let D be a difference operator such that $Df(u) = f(u+2)D$ for any function $f(u)$. We shall use notation: $\overrightarrow{\prod_{k=1}^{m}} g_k = g_1 g_2 \cdots g_m$ and $\overleftarrow{\prod_{k=1}^{m}} g_k = g_m g_{m-1} \cdots g_1$. By using the variables (2.8)–(2.11), we introduce a factorized difference L operator for each algebra.

For the $U_q\left(A_{2n}^{(2)}\right)$ case:

$$\begin{aligned}
L(u) &= \prod_{a=1}^{\overrightarrow{n}}(1 - z_{\bar{a}}(u)D)(1 - z_0(u)D)\prod_{a=1}^{\overleftarrow{n}}(1 - z_a(u)D) \\
&= \prod_{a=1}^{\overleftarrow{2n+1}}(1 - x_a(u)D).
\end{aligned} \tag{2.12}$$

For the $U_q\left(A_{2n-1}^{(2)}\right)$ case:

$$L(u) = \prod_{a=1}^{\overrightarrow{n}}(1 - z_{\bar{a}}(u)D)\prod_{a=1}^{\overleftarrow{n}}(1 - z_a(u)D) = \prod_{a=1}^{\overleftarrow{2n}}(1 - x_a(u)D). \tag{2.13}$$

For the $U_q\big(D^{(2)}_{n+1}\big)$ case:

$$L(u) = \overrightarrow{\prod_{a=1}^{n+1}}(1 - z_{\bar{a}}(u)D)\left(1 - z_{n+1}(u)z_{\overline{n+1}}(u+2)D^2\right)^{-1}\overleftarrow{\prod_{a=1}^{n+1}}(1 - z_a(u)D). \tag{2.14}$$

For the $U_q\big(D^{(3)}_4\big)$ case:

$$L(u) = \overrightarrow{\prod_{a=1}^{4}}(1 - z_{\bar{a}}(u)D)\left(1 - z_4(u)z_{\bar{4}}(u+2)D^2\right)^{-1}\overleftarrow{\prod_{a=1}^{4}}(1 - z_a(u)D). \tag{2.15}$$

In general, $L(u)$ (2.12)–(2.15) are power series of D whose coefficients lie in $\mathbb{Z}\big[Y_a(u)^{\pm 1}\big]_{a\in I_\sigma;u\in\mathbb{C}}$. We assume $\mathcal{S}_a$ acts on these coefficients linearly.

Proposition 2.1. *For $a \in I_\sigma$, we have $(\mathcal{S}_a \cdot L)(u) = 0$.*

The proof is similar to the non-twisted case [4]. So we just mention the lemmas which are necessary to the $U_q\big(D^{(3)}_4\big)$ case.

Lemma 2.2. *For the $U_q\big(D^{(3)}_4\big)$ case, let*

$$H_1(u) = Y_1(u) + \frac{Y_2^3(u+1)}{Y_1(u+2)} \qquad H_2(u) = Y_2^3(u) + \frac{Y_1^3(u+1)}{Y_2^3(u+2)}$$

$$K_1(u) = \frac{1}{Y_1(u)} + \frac{Y_1(u-2)}{Y_2^3(u-1)} \qquad K_2(u) = \frac{1}{Y_2^3(u)} + \frac{Y_2^3(u-2)}{Y_1^3(u-1)}$$

then $(\mathcal{S}_a \cdot H_a)(u) = (\mathcal{S}_a \cdot K_a)(u) = 0$ for $a = 1, 2$.

Lemma 2.3. *For the $U_q\big(D^{(3)}_4\big)$ case, one can rewrite $L(u)$ (2.15) as follows:*

$$L(u) = \left(1 - K_1(u+7)D + \frac{1}{Y_2^3(u+8)}D^2\right)\left(1 - \sum_{j=0}^{\infty} A_j(u)D^{2j+1} + \sum_{j=0}^{\infty} B_j(u)D^{2j+2}\right)$$
$$\times\left(1 - H_1(u+1)D + Y_2^3(u+2)D^2\right)$$

where

$$A_j(u) = K_1\left(u + 4j + 5 + \frac{\pi \mathrm{i}}{3\hbar}\right) H_1\left(u + 3 - \frac{\pi \mathrm{i}}{3\hbar}\right)$$
$$+ (1 - \delta_{j0})K_1\left(u + 4j + 5 - \frac{\pi \mathrm{i}}{3\hbar}\right) H_1\left(u + 3 + \frac{\pi \mathrm{i}}{3\hbar}\right)$$
$$B_j(u) = K_1\left(u + 4j + 7 + \frac{\pi \mathrm{i}}{3\hbar}\right) H_1\left(u + 3 + \frac{\pi \mathrm{i}}{3\hbar}\right)$$
$$+ K_1\left(u + 4j + 7 - \frac{\pi \mathrm{i}}{3\hbar}\right) H_1\left(u + 3 - \frac{\pi \mathrm{i}}{3\hbar}\right) - \delta_{j0}\frac{Y_2^3(u+4)}{Y_2^3(u+6)}.$$

Lemma 2.4. *For the $U_q\big(D_4^{(3)}\big)$ case, one can expand the Y_2 dependent part in $L(u)$ (2.15):*

$$(1-z_{\bar{2}}(u)D)(1-z_{\bar{3}}(u)D) = 1 - Y_1(u+5)K_2(u+6)D + \frac{Y_1(u+5)Y_1(u+7)}{Y_1^3(u+7)}D^2$$

$$(1-z_3(u)D)(1-z_2(u)D) = 1 - \frac{H_2(u+2)}{Y_1(u+3)}D + \frac{Y_1^3(u+3)}{Y_1(u+3)Y_1(u+5)}D^2.$$

We shall expand $L(u)$ as

$$L(u) = \sum_{a=0}^{\infty}(-1)^a T^a(u+a)D^a. \tag{2.16}$$

In particular, we have $T^0(u) = 1$ and $T^a(u) = 0$ for $a \in \mathbb{Z}_{<0}$. For the $U_q\big(A_N^{(2)}\big)$ case, (2.16) becomes a polynomial in D of order $N+1$ and $T^a(u) = 0$ for $a \in \mathbb{Z}_{\geqslant N+2}$.

Remark 2.5. There is a homomorphism β analogous to that in [5]:

$$\beta : \mathbb{Z}\left[Y_a(u)^{\pm 1}\right]_{a\in I_\sigma; u\in\mathbb{C}} \to \mathbb{Z}\left[\mathrm{e}^{\pm\frac{1}{r_a}\Lambda_a}\right]_{a\in I_\sigma} \qquad \beta\left(Y_a(u)^{\pm 1}\right) = \mathrm{e}^{\pm\frac{1}{r_a}\Lambda_a}$$

where $\{\Lambda_a\}_{a\in I_\sigma}$ are the fundamental weights of a rank n subalgebra $\mathring{\mathfrak{g}}$ of $X_N^{(r)}$: $\big(X_N^{(r)}, \mathring{\mathfrak{g}}\big) = \big(X_n^{(1)}, X_n\big), \big(A_{2n}^{(2)}, C_n\big), \big(A_{2n-1}^{(2)}, C_n\big), \big(D_{n+1}^{(2)}, B_n\big), \big(D_4^{(3)}, G_2\big), \big(E_6^{(2)}, F_4\big)$. Note that the image of β is independent of the parameter $\hbar$. In particular, $\beta(T^a(u)) \in \mathbb{Z}[\mathrm{e}^{\pm\Lambda_b}]_{b\in I_\sigma}$ is a linear combination of $\mathring{\mathfrak{g}}$ characters (cf section 6 in [18]). For $1 \leqslant a \leqslant b$ $\big(U_q\big(A_{2n}^{(2)}\big), U_q\big(A_{2n-1}^{(2)}\big)$: $b = n$; $U_q\big(D_{n+1}^{(2)}\big)$: $b = n-1$; $U_q\big(D_4^{(3)}\big)$: $b = 2\big)$, $T^a(u)$ contains a term $Y_a^{r_a}(u) = \prod_{k=1}^{a} z_k(u+a-2k)$: $\beta\big(Y_a^{r_a}(u)\big) = \mathrm{e}^{\Lambda_a}$. In the context of the analytic Bethe ansatz [9] (resp. the theory of q-characters [5]), $Y_a^{r_a}(u)$ corresponds to the top term of DVF (resp. the highest weight monomial of the q-character) for the Kirillov–Reshetikhin module $W_1^{(a)}(u)$ over $U_q\big(X_N^{(r)}\big)$.

From proposition 2.1, we obtain:

Corollary 2.6. *For $a \in I_\sigma$ and $b \in \mathbb{Z}$, we have $\big(\mathcal{S}_a \cdot T^b\big)(u) = 0$.*

For the $U_q\big(A_N^{(2)}\big)$ case, there is a duality among $\{T^a(u)\}_{a\in\mathbb{Z}; u\in\mathbb{C}}$.

Proposition 2.7. *For the $U_q\big(A_N^{(2)}\big)$ case, we have*

$$T^a(u) = T^{N+1-a}\left(u + \frac{\pi \mathrm{i}}{2\hbar}\right) \qquad a \in \mathbb{Z}.$$

This relation is given in [1] as 'modulo σ relation'. The proof of this proposition is similar to the $B^{(1)}(0|n)$ case [16], which corresponds to $N = 2n$ and $\frac{\pi \mathrm{i}}{\hbar} \to 0$.

One can show

$$L(u)Q_1^{r_1}(u) = 0. \tag{2.17}$$

A T–Q relation follows from (2.17):

$$\sum_{a=0}^{\infty}(-1)^a T^a(u+a)Q_1^{r_1}(u+2a) = 0. \tag{2.18}$$

We shall expand $L(u)^{-1}$ as

$$L(u)^{-1} = \sum_{m=0}^{\infty} T_m(u+m)D^m. \tag{2.19}$$

In particular, we have $T_0(u) = 1$ and $T_m(u) = 0$ for $m \in \mathbb{Z}_{<0}$. From the relation $L(u)L(u)^{-1} = 1$, we obtain a T–T relation

$$\sum_{a=0}^{m}(-1)^a T_{m-a}(u+m+a)T^a(u+a) = \delta_{m0}. \tag{2.20}$$

From the relation $L(u)^{-1}L(u) = 1$, we also have

$$\sum_{a=0}^{m}(-1)^a T_{m-a}(u-m-a)T^a(u-a) = \delta_{m0}. \tag{2.21}$$

In particular for the $U_q\big(A_N^{(2)}\big)$ case, the T–Q relation (2.18) reduces to

$$\sum_{a=0}^{N+1}(-1)^a T^a(u+a)Q_1(u+2a) = 0. \tag{2.22}$$

From the proposition 2.7, one can rewrite this as follows:

$$\sum_{a=0}^{N+1}(-1)^a T^a(u-a)Q_1\left(u-2a+g+\frac{\pi \mathrm{i}}{2\hbar}\right) = 0 \tag{2.23}$$

where $g = N+1$ is the dual Coxeter number of $A_N^{(2)}$. If one assumes $\lim_{m\to\infty} T_m(u+m)$ (resp. $\lim_{m\to\infty} T_m(u-m)$) is proportional to $Q_1(u)$ (resp. $Q_1(u+g+\frac{\pi \mathrm{i}}{2\hbar})$), then one can recover the T–Q relation (2.22) (resp. (2.23)) from the T–T relation (2.20) (resp. (2.21)).

3. Solution of the *T*-system

The goal of this section is to give a Casorati determinant solution to the $U_q\big(A_N^{(2)}\big)$ T-system (1.1), (1.2). Consider the following difference equation:

$$L(u)w(u) = 0 \tag{3.1}$$

where $L(u)$ is the difference L operator (2.12) and (2.13) for $U_q\big(A_N^{(2)}\big)$. By using a basis $\{w_1(u), w_2(u), \ldots, w_{N+1}(u)\}$ of the solutions of (3.1), we define a Casorati determinant:

$$[i_1, i_2, \ldots, i_{N+1}] = \begin{vmatrix} w_1(u+2i_1) & w_1(u+2i_2) & \cdots & w_1(u+2i_{N+1}) \\ w_2(u+2i_1) & w_2(u+2i_2) & \cdots & w_2(u+2i_{N+1}) \\ \vdots & \vdots & \ddots & \vdots \\ w_{N+1}(u+2i_1) & w_{N+1}(u+2i_2) & \cdots & w_{N+1}(u+2i_{N+1}) \end{vmatrix}.$$

Setting $w = w_1, w_2, \ldots, w_{N+1}$ in (3.1) and noting the relation $T^{N+1}(u) = 1$, we obtain the following relation:

$$[0, 1, \ldots, N] = [1, 2, \ldots, N+1]. \tag{3.2}$$

Owing to Cramer's formula, we also have:

Proposition 3.1. *For $a \in \{0, 1, \ldots, N+1\}$, we have*

$$T^a(u+a) = \frac{[0, 1, \ldots, a-1, a+1, \ldots, N+1]}{[0, 1, \ldots, N]}.$$

Lemma 3.2. *For the $U_q\big(A_N^{(2)}\big)$ case, one can rewrite $L(u)$ (2.12), (2.13) as*

$$L(u) = \prod_{a=1}^{\overrightarrow{N+1}}\left(x_a\left(u+N+1-2a+\frac{\pi \mathrm{i}}{2\hbar}\right) - D\right).$$

Let $\xi_m^{(a)}(u) = [0, 1, \ldots, a-1, a+m, a+m+1, \ldots, N+m]$ and $\xi(u) = \xi_0^{(1)}(u) = [0, 1, \ldots, N]$. Note that $\xi_m^{(0)}(u) = \xi(u)$ follows from (3.2). For $1 \leqslant a \leqslant N+1$, we introduce a difference operator

$$L_a(u) = \overrightarrow{\prod_{b=N+2-a}^{N+1}} \left(D - x_b\left(u + N + 1 - 2b + \frac{\pi \mathrm{i}}{2\hbar}\right)\right). \tag{3.3}$$

In particular we have $L_{N+1}(u) = (-1)^{N+1} L(u)$. We choose a basis of the solutions of (3.1) so that it satisfies $L_a(u) w_b(u) = 0$ for $1 \leqslant b \leqslant a \leqslant N+1$: $w_a \in \mathrm{Ker} L_a$. For this basis, the following lemma holds.

Lemma 3.3. *Let $\{i_k\}$ be integers such that $0 = i_0 < i_1 < \cdots < i_N$, $\mu = (\mu_k)$ the Young diagram whose kth row is $\mu_k = i_{N+1-k} + k - N - 1$, and $\mu' = (\mu'_k)$ the transposition of μ. We assign coordinates $(j,k) \in \mathbb{Z}^2$ on the skew-Young diagram $(\mu_1^{N+1})/\mu$ such that the row index j increases as we go upwards and the column index k increases as we go from left to right and that $(1,1)$ is on the bottom left corner of $(\mu_1^{N+1})/\mu$:*

$$\frac{[i_0, i_1, \ldots, i_N]}{[0, 1, \ldots, N]} = \sum_b \prod_{(j,k)\in(\mu_1^{N+1})/\mu} x_{b(j,k)}(u + 2j + 2k - 4)$$
$$= \det_{1\leqslant j,k\leqslant \mu_1}\left(\mathcal{T}^{\mu'_j - j + k}\left(u + N - 1 + j + k - \mu'_j + \frac{\pi \mathrm{i}}{2\hbar}\right)\right)$$

where the summation is taken over the semi-standard tableau b on the skew-Young diagram $(\mu_1^{N+1})/\mu$ as the set of elements $b(j,k) \in \{1, 2, \ldots, N+1\}$ labelled by the coordinates (j,k) mentioned above.

The proof is similar to the $U_q(C_n^{(1)})$ case [4], where we use a theorem in [19] and proposition 2.7. Note that lemma 3.3 reduces to proposition 3.1 if we set $i_b = b$ for $0 \leqslant b \leqslant a-1$ and $i_b = b+1$ for $a \leqslant b \leqslant N$. From proposition 2.7 and lemma 3.3, one can show:

Lemma 3.4. *For $a \in \{0, 1, \ldots, N+1\}$, we have*

$$\frac{\xi_m^{(a)}(u)}{\xi(u)} = \frac{\xi_m^{(N-a+1)}\left(u + 2a - N - 1 + \frac{\pi \mathrm{i}}{2\hbar}\right)}{\xi\left(u + 2a - N - 1 + \frac{\pi \mathrm{i}}{2\hbar}\right)}.$$

The following relation is a kind of Hirota–Miwa equation [2, 3], which is a Plücker relation and used in a similar context [4, 20–22].

Lemma 3.5. $\xi_m^{(a)}(u)\xi_m^{(a)}(u+2) = \xi_{m-1}^{(a)}(u)\xi_{m+1}^{(a)}(u+2) + \xi_m^{(a-1)}(u)\xi_m^{(a+1)}(u+2)$.

From lemmas 3.4 and 3.5, we finally obtain:

Theorem 3.6. *For $a \in I_\sigma$ and $m \in \mathbb{Z}_{\geqslant 1}$,*

$$T_m^{(a)}(u) = \frac{\xi_m^{(a)}(u - a - m + 1)}{\xi(u - a - m + 1)}$$

satisfies the T-system for $U_q(A_N^{(2)})$ (1.1), (1.2).

There is another expression of the solution to the $U_q(A_N^{(2)})$ T-system (1.1), (1.2), which follows from a reduction of Bazhanov and Reshetikhin's Jacobi–Trudi type formula [13] (cf section 5 in [1])

$$T_m^{(a)}(u) = \det_{1\leqslant j,k\leqslant m}(\mathcal{T}^{a-j+k}(u + j + k - m - 1)) \tag{3.4}$$

where $\mathcal{T}^a(u)$ obeys the following condition:

$$\mathcal{T}^a(u) = \begin{cases} 0 & \text{if} \quad a < 0 \text{ or } a > N+1 \\ 1 & \text{if} \quad a = 0 \text{ or } a = N+1 \\ T_1^{(a)}(u) & \text{if} \quad 1 \leqslant a \leqslant n \\ T_1^{(N-a+1)}\left(u + \frac{\pi \mathrm{i}}{2\hbar}\right) & \text{if} \quad n+1 \leqslant a \leqslant N. \end{cases} \tag{3.5}$$

Through the identification $\mathcal{T}^a(u) = T^a(u)$ and lemma 3.3, (3.4) reproduces the solution in theorem 3.6, and also the tableaux sum expression in [1].

4. Discussion

In this paper, we have dealt with the T-system without the vacuum part. On applying our results to realistic problems in solvable lattice models or integrable field theories, we must specify the Baxter Q-function, and recover the vacuum part whose shape depends on each model. We can easily recover the vacuum part multiplying the vacuum function $\psi_a(u)$ by the function $z_a(u)$ so that $\psi_a(u)$ is compatible with the Bethe ansatz equation of the form (cf [17, 23])

$$\Psi_a\left(u_j^{(a)}\right) = \prod_{b=1}^{n'} \frac{Q_b^{r_{ab}}\left(u_j^{(a)} + (\alpha_a|\alpha_b)\right)}{Q_b^{r_{ab}}\left(u_j^{(a)} - (\alpha_a|\alpha_b)\right)} \qquad a \in I_\sigma. \tag{4.1}$$

In the case of the solvable vertex model, it was conjectured [23] that $\Psi_a(u)$ is given as a ratio of Drinfeld polynomials.

A remarkable connection between DVF and the q-character was pointed out in [5]. It was also conjectured [4] that q-characters of Kirillov–Reshetikhin modules over $U_q\left(X_n^{(1)}\right)$ satisfy the T-system [24]. It is natural to expect that similar phenomena are also observed for the twisted case $U_q\left(X_N^{(r)}\right)$ $(r > 1)$. Thus one may look upon $T_m^{(a)}(u)$ in theorem 3.6 (or $T^a(u)$) as a kind of q-character. Precisely speaking, in view of a correspondence [25] between DVF and generators of the deformed W-algebra, one may need to slightly modify $T_m^{(a)}(u)$ (or $T^a(u)$) (in particular, the factor $\frac{\pi \mathrm{i}}{\hbar}$) to identify $T_m^{(a)}(u)$ (or $T^a(u)$) with the q-character of the Kirillov–Reshetikhin module over $U_q\left(X_N^{(r)}\right)$ $(r > 1)$.

We can also easily construct difference L operators associated with superalgebras by using the results on the analytic Bethe ansatz [16, 26–28]. However, their orders are infinite as $U_q\left(B_n^{(1)}\right)$, $U_q\left(D_n^{(1)}\right)$, $U_q\left(D_{n+1}^{(2)}\right)$, $U_q\left(D_4^{(3)}\right)$ cases. Thus we will need some new ideas to construct Casorati determinant-like solutions to the T-system for superalgebras.

Acknowledgments

The author would like to thank Professor A Kuniba for explaining the results on [4]. He is financially supported by Inoue Foundation for Science.

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